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WEIBULL ACCELERATED LIFE TESTS WHEN THERE ARE COMPETING CAUSES —ETC(U)

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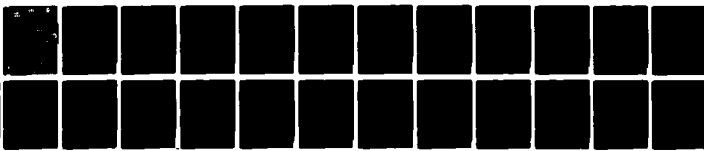
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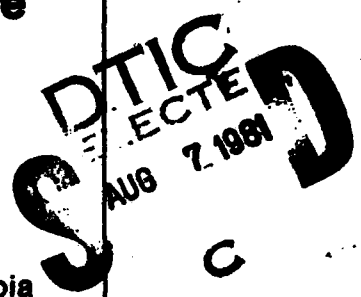
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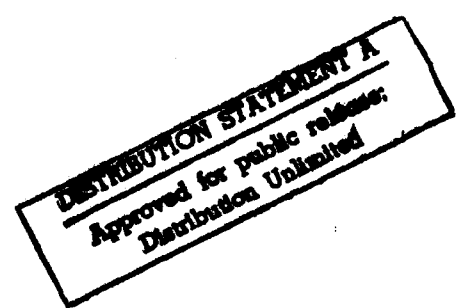
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Maximum likelihood estimators of the component parameters are obtained for type I, type II, and progressive censoring. These estimators are used to obtain estimators of the probability of component and system survival under normal operating conditions. This method is illustrated by an example.

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WEIBULL ACCELERATED LIFE TESTS WHEN THERE ARE
COMPETING CAUSES OF FAILURE

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Key Words and Phrases: safe dose levels, competing risks,
accelerated lifetests, Hartley and Sielkin model.

ABSTRACT

Accelerated life testing of a product under more severe than normal conditions is commonly used to reduce test time and costs. Data collected at such accelerated conditions are used to obtain estimates of the parameters of a stress translation function. This function is then used to make inference about the product's life under normal operating conditions.

We consider the problem of accelerated life tests when the product of interest is a p component series system. Each of the components is assumed to have an independent Weibull time to failure distribution with different shape parameters and different scale parameters which are increasing functions

of the stress. A general model is used for the scale parameter which includes the standard engineering models as special cases. This model also has an appealing biological interpretation.

Maximum likelihood estimators of the component parameters are obtained for type I, type II, and progressive censoring. These estimators are used to obtain estimators of the probability of component and system survival under normal operating conditions. This method is illustrated by an example.

1. INTRODUCTION

Accelerated life testing of a product is often used to obtain information on its performance under normal use conditions. Such testing involves subjecting test items to conditions more severe than encountered in the item's everyday use. This results in decreasing the item's mean life and leads to shorter test times and reduced experimental costs. In engineering applications accelerated conditions are produced by testing items at higher than normal temperature, voltage, pressure, load, etc. In biological applications accelerated conditions arise when large doses of a chemical or radiological agent are given. In both cases the data collected at the high stresses is used to extrapolate to some low stress level where testing is not feasible.

Several authors have considered the problem of analyzing accelerated life tests when the product has only a single mode of failure. Nelson (1974a) has a bibliography of applications and analysis of each tests. Mann, Schafer, and Singpurwalla (1974) derive least squares and maximum likelihood estimators of model parameters when the underlying failure distribution is exponential. Nelson (1972c) describes a graphical solution to this problem.

Nelson (1970) derives graphical, maximum likelihood and least squares estimators of model parameters when the underlying distribution is Weibull. Meeker and Nelson (1974a and 1974b) derive maximum likelihood estimators of the model parameters in the Weibull case when the data is type I or type II censored. They also discuss the optimal strategy for designing such tests. Mann (1972) has also discussed optimal design strategy. Tolerance bounds for the Weibull model are discussed in Mann (1978).

Several papers have been written on analyzing accelerated life tests when the failure distribution is normal or log normal. In a series of papers Nelson (1971, 1972a and 1972b) has considered maximum likelihood, least squares, and graphical estimation procedures for an Arrhenius model when all failure times are known. For this model it is assumed that mean of the log failure time is linear in the stress, and that the variance is independent of the stress. Nelson and Hahn (1972, 1973) derive best linear unbiased estimators of the regression parameters of this model for type II censored samples. Kielpinski and Nelson (1975) discuss maximum likelihood estimation procedures for this model when the sample is type I censored.

Several papers have been written on analyzing accelerated life tests when more than one failure mode is present. Here failures can occur from any one of k independent causes. Sample information consists of a failure time for each item and the cause of failure. Assuming that for a given stress V each failure mode follows an independent log normal distribution with parameters $\mu_i(V) = \alpha_i + \beta_i V$, and σ_i^2 constant with respect to V , $i = 1, \dots, k$, Nelson (1973) obtains graphical estimates of α_i and β_i when there is no censoring. For this model, Nelson (1974b) obtains maximum likelihood estimates of α_i , β_i and σ_i^2 .

Klein and Basu (1980a) have considered the above problem when the component lifetimes are exponentially distributed and the data is type I, type II or progressively censored. Klein and Basu (1980b) have described the analysis of accelerated life tests of series systems when the component failure times follow a Weibull distribution with common shape parameter. The object of this paper is to consider the case when the shape parameters are different.

In section 2 we present the model to be used for accelerated life tests in the competing risks framework. In section 3 we use this model to analyze accelerated life tests where there are competing causes of failure and the data is type I, type II, or progressively censored. Finally, in section 4 an example is presented.

2. THE MODEL

The problem considered in the sequel is as follows. Consider a p component system with component lifetimes X_1, X_2, \dots, X_p . Suppose that under normal stress conditions these components have long lifetimes making testing at such conditions unfeasible. To reduce test time and cost, s stresses, V_1, \dots, V_s are selected and a life test is conducted at constant application of the selected stress. We wish to use this information to make inference about the component lifetimes under normal stress conditions.

Consider the following model introduced by Klein and Basu (1980b) elsewhere.

At a stress V_i , $i = 1, \dots, s$ assume that the j^{th} component has a hazard rate given by

$$h_j(x, V_i; \underline{\alpha}_j, \underline{\beta}_j) = g_j(x, \underline{\alpha}_j) \lambda_j(V_i, \underline{\beta}_j) \quad (2.1)$$

$$i = 1, \dots, s \quad j = 1, \dots, p.$$

For $g_j(x, \underline{\alpha}_j)$ a Weibull form is assumed, that is

$$g_j(x, \alpha_j) = \alpha_j t^{\alpha_j - 1}, \alpha_j \geq 0, t > 0. \quad (2.2)$$

The α_j 's may vary from component to component to allow for differences in component reliability.

For $\lambda_j(V, \underline{\beta}_j)$ we assume a model of the form

$$\lambda_j(V, \underline{\beta}_j) = \exp \left(\sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V) \right). \quad (2.3)$$

where $\theta_{j0}(V) = 1$ and $\theta_{j1}(V), \dots, \theta_{jk_j}(V)$ are k_j non-decreasing functions of V . The $\theta_j(\cdot)$'s may differ from one component to another.

This model includes the standard models, namely, the power rule with $\lambda_j(V, \underline{\beta}_j) = \beta_{j0} V^{\beta_{j1}}$; the Arrhenius reaction rate model with $\lambda_j(V, \underline{\beta}_j) = \exp(\beta_{j0} - \beta_{j1}/V)$; and the Eyring model for a single stress with $\lambda_j(V, \underline{\beta}_j) = V^{\beta_{j1}} \exp(\beta_{j0} - \beta_{j2}/V)$ as special cases.

The model also can be derived from the interpretation of the effects of a carcinogen on a cell as proposed by Armitage and Doll (1961). For details see Klein and Basu (1980b). To produce cancer in a single cell, k independent events must occur. The effect of an increased dose of a carcinogen is to increase the rate at which these k events occur. If, for the j^{th} disease, this increase is of the form $\exp(\beta_{j\ell} \theta_{j\ell}(V))$ for $\ell = 1, \dots, k_j$ the model (2.3) is obtained. If this increase is assumed linear the model of Hartley and Sielkin (1977) is obtained. Thus the model of Hartley and Sielkin is a first order Taylor Series approximation to (2.3) when $\theta_{j\ell}(V) = V$ for $\ell = 1, \dots, k_j$.

Consider an accelerated life test conducted at constant applications of s stress level, V_1, \dots, V_s . Let $X_{i1}, X_{i2}, \dots, X_{ip}$ denote the component lifetimes of the p component

series system put on test at stress V_i . Assume that the component lifetimes are independent. We are not allowed to observe X_{i1}, \dots, X_{ip} directly but, instead, we observe $Y_i = \text{minimum}(X_{i1}, \dots, X_{ip})$ and an indicator variable which describes which of the p components is the minimum. We shall use the method of maximum likelihood to estimate α_j and $\beta_j = (\beta_{j0}, \dots, \beta_{jk_j})$, $j = 1, \dots, p$ for various censoring schemes.

3. ESTIMATION OF PARAMETERS

3.1 Type I censoring

For this censoring scheme n_i items are put on test at stress V_i , $i = 1, \dots, s$. The ℓ^{th} system on test at stress V_i is tested until it fails or until some fixed time $\tau_{i\ell}$ at which it is removed from the study. The $\tau_{i\ell}$'s may vary from item to item to allow for staggered entry into the study. Let r_i be the number of systems which fail prior to their censoring time at stress V_i . Let r_{ij} denote the number of these whose failure was caused by failure of the j^{th} component. Let $X_{ij\ell}$ denote the failure time of the r_{ij} systems whose failure is due to failure of component j . For convenience let $Y_{i\ell}$, $i = 1, \dots, s$, $\ell = 1, \dots, r_i$ denote the failure times of the r_i systems regardless of the cause of failure.

The overall log likelihood can be written as

$$\ln L = \sum_{j=1}^p \ln L_j \quad (3.1.1)$$

where

$$\begin{aligned} \ln L_j = & \sum_{i=1}^s r_{ij} \left(\sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right) + r_{ij} \ln \alpha_j + (\alpha_j - 1) \sum_{\ell=1}^{r_{ij}} \ln X_{ij\ell} \\ & - T_i(\alpha_j) \exp \left(\sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right), \quad j = 1, \dots, p \end{aligned} \quad (3.1.2)$$

where

$$T_i(\alpha_j) = \sum_{\ell=1}^{r_i} Y_i^{\alpha_j} + \sum_{\ell=1}^{n_i-r_i} \tau_{i\ell}^{\alpha_j}, \quad i = 1, \dots, s \quad (3.1.3)$$

with $\tau_{i\ell}^{\alpha_j}$, $i = 1, \dots, s$, $\ell = 1, \dots, n_i - r_i$ the censoring times of the $n_i - r_i$ systems removed from the accelerated life test. When all items on test at stress V_i have a common censoring

time, τ_i , then $T_i(\alpha_j) = \sum_{\ell=1}^{r_i} Y_i^{\alpha_j} + (n_i - r_i)\tau_i$.

The likelihood equations, which must be solved numerically for the maximum likelihood estimators, $\hat{\alpha}_j$, $\hat{\beta}_{j0}$, $\hat{\beta}_{j1}$, ..., $\hat{\beta}_{jk_j}$, are:

$$0 = \frac{\delta \ln L_j}{\delta \alpha_j} = \sum_{i=1}^s \left[\frac{r_{ij}}{\alpha_j} + \sum_{\ell=1}^{r_{ij}} \ln X_{ij\ell} - T_i^{(1)}(\alpha_j) \exp \left(\sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right) \right],$$

$$j = 1, \dots, p \quad (3.1.4)$$

where

$$T_i^{(1)}(\alpha_j) = \sum_{\ell=1}^{r_i} Y_{i\ell}^{\alpha_j} \ln Y_{i\ell} + \sum_{\ell=1}^{n_i-r_i} \tau_{i\ell}^{\alpha_j} \ln \tau_{i\ell}, \quad i = 1, \dots, s, \quad (3.1.5)$$

and

$$0 = \frac{\delta \ln L_j}{\delta \beta_{ju}} = \sum_{i=1}^s r_{ij} \theta_{ju}(V_i) - T_i(\alpha_j) \theta_{ju}(V_i) \exp \left(\sum_{\ell=0}^{k_j} \beta_{j\ell} \theta_{j\ell}(V_i) \right),$$

$$j = 1, \dots, p \quad u = 0, \dots, k_j \quad (3.1.6)$$

The second partial derivatives of the log likelihood are

$$-\frac{\delta^2 \ln L_j}{\delta \alpha_j^2} = \sum_{i=1}^s \frac{r_{ij}}{\alpha_j^2} - \lambda_{ij} T_i^{(2)}(\alpha_j), \quad j = 1, \dots, p. \quad (3.1.7)$$

$$T_i^{(2)}(\alpha_j) = \sum_{\ell=1}^{r_i} Y_{i\ell}^{\alpha_j} (\ln Y_{i\ell})^2 +$$

$$\sum_{\ell=1}^{n_i - r_i} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell})^2, \quad i = 1, \dots, p, \quad (3.1.8)$$

$$-\frac{\delta^2 \ln L_j}{\delta \alpha_j \delta \beta_{ju}} = \sum_{i=1}^s \lambda_{ij} \theta_{ju}(V_i) T_i^{(1)}(\alpha_j)$$

$$j = 1, \dots, p \quad u = 0, \dots, k_j \quad (3.1.9)$$

and

$$\frac{\delta^2 \ln L_j}{\delta \beta_{ju} \delta \beta_{jw}} = \sum_{i=1}^s \lambda_{ij} T_i(\alpha_j) \theta_{jw}(V_i) \theta_{ju}(V_i)$$

$$w = 1, \dots, k_j \quad u = 0, \dots, k_j. \quad (3.1.10)$$

To find the information matrix let

$$C_{i\ell} = \begin{cases} 1 & \text{if } Y_{i\ell} \leq \tau_{i\ell}, \quad i = 1, \dots, s, \ell = 1, \dots, n_i \\ 0 & \text{otherwise} \end{cases} \quad (3.1.11)$$

and define

$$\eta_{ij} = P(C_{i\ell} = 1) = 1 - \exp \left(- \sum_{j=1}^p \lambda_{ij} \tau_{i\ell}^{\alpha_j} \right),$$

$$i = 1, \dots, s \quad \ell = 1, \dots, n_i. \quad (3.1.12)$$

The conditional density function of $Y_{i\ell}$ given $C_{i\ell} = 1$ is

$$f(y_{i\ell} | C_{i\ell} = 1) = \begin{cases} \frac{\sum_{j=1}^p \alpha_j \lambda_{ij} y^{\alpha_j - 1}}{n_{i\ell}} \exp \left(- \sum_{j=1}^p \lambda_{ij} y^{\alpha_j} \right), & y < \tau_{i\ell} \\ 0 & \text{otherwise.} \end{cases} \quad (3.1.13)$$

Now

$$\begin{aligned} E(T_i(\alpha_j)) &= E \left(\sum_{\ell=1}^{n_i} C_{i\ell} Y_{i\ell}^{\alpha_j} + \sum_{\ell=1}^{n_i} (1 - C_{i\ell}) \tau_{i\ell}^{\alpha_j} \right) \\ &= \sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} y^{\alpha_j} \left(\sum_{m=1}^p \alpha_m \lambda_{im} y^{\alpha_m - 1} \right) \exp \left(- \sum_{m=1}^p \lambda_{im} y^{\alpha_m} \right) dy \\ &\quad + \sum_{\ell=1}^{n_i} \tau_{i\ell}^{\alpha_j} \exp \left(- \sum_{m=1}^p \lambda_{im} \tau_{i\ell}^{\alpha_m} \right) \end{aligned}$$

$$i = 1, \dots, p \quad j = 1, \dots, s. \quad (3.1.14)$$

where the integral must be evaluated numerically. Similarly,

$$\begin{aligned} E(T_i^{(1)}(\alpha_j)) &= \\ &\sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} y^{\alpha_j} \ln y \left(\sum_{m=1}^p \alpha_m \lambda_{im} y^{\alpha_m - 1} \right) \exp \left(- \sum_{m=1}^p \lambda_{im} y^{\alpha_m} \right) dy. \\ &\quad + \sum_{\ell=1}^{n_i} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell}) \exp \left(- \sum_{m=1}^p \lambda_{im} \tau_{i\ell}^{\alpha_m} \right), \end{aligned} \quad (3.1.15)$$

and

$$E(T^{(2)}(\alpha_j)) =$$

$$\sum_{\ell=1}^{n_i} \int_0^{\tau_{i\ell}} y^{\alpha_j} (\ln y)^2 \left(\sum_{m=1}^p \alpha_m \lambda_{im} y^{\alpha_m-1} \right) \exp \left(- \sum_{m=1}^p \lambda_{im} y^{\alpha_m} \right) dy$$

$$+ \sum_{\ell=1}^{n_i} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell})^2 \exp \left(- \sum_{m=1}^p \lambda_{ij} \tau_{i\ell}^{\alpha_m} \right),$$

$$i = 1, \dots, s \quad j = 1, \dots, p. \quad (3.1.16)$$

Now the probability that the ℓ^{th} system fails prior to time $\tau_{i\ell}$ due to failure of cause j at stress i for $i = 1, \dots, s$, $j = 1, \dots, p$, $\ell = 1, \dots, n_i$ is

$$\psi_{ij\ell} = \int_0^{\tau_{i\ell}} \lambda_{ij} u^{\alpha_j-1} \exp \left(- \sum_{m=1}^p \lambda_{im} u^{\alpha_m} \right) du \quad (3.1.17)$$

which must be evaluated numerically. Hence

$$E(r_{ij}) = \sum_{\ell=1}^{n_i} \psi_{ij\ell}. \quad (3.1.18)$$

The asymptotic covariance matrix can now be obtained by using (3.1.18), (3.1.16), (3.1.15), and (3.1.14) to calculate the expected values of (3.1.7), (3.1.9), and (3.1.10). An estimator of this matrix can be obtained by substituting $\hat{\alpha}_j$, and $\hat{\beta}_j$ in the appropriate expressions.

3.2 Type II censoring.

For this censoring scheme n_i systems are put on test at each of the s stress levels and testing continues until a preassigned number r_i have failed at which time testing is stopped. Suppose that r_{ij} systems fail due to failure of the j^{th} component, $j = 1, \dots, p$. Let $X_{ij1}, \dots, X_{ijr_{ij}}$ denote the failure time of those r_{ij} systems at stress V_i whose failure was caused by

failure of the j th component, $i = 1, \dots, s$, $j = 1, \dots, p$, $\ell = 1, \dots, r_i$. Let $Y_{i(1)}, \dots, Y_{i(r_i)}$ denote the ordered failure times of the r_i systems observed to fail at stress i , regardless of the mode of failure.

One can show that the likelihood of interest is given by (3.1.1) and (3.1.2) with

$$T_i(\alpha_j) = \sum_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha_j} + (n_i - r_i) Y_{i(r_i)}^{\alpha_j},$$

$$i = 1, \dots, s \quad j = 1, \dots, p. \quad (3.2.1)$$

The likelihood equations are given by (3.1.4) and (3.1.5) with

$$T_i^{(1)}(\alpha_j) = \sum_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha_j} \ln Y_{i(\ell)} + (n_i - r_i) Y_{i(r_i)}^{\alpha_j} \ln Y_{i(r_i)},$$

$$i = 1, \dots, s \quad j = 1, \dots, p. \quad (3.2.2)$$

The matrix of second partial derivatives is given by (3.1.7) to (3.1.9) with

$$T_i^{(2)}(\alpha_j) = \sum_{\ell=1}^{r_i} Y_{i(\ell)}^{\alpha_j} (\ln Y_{i(\ell)})^2 + (n_i - r_i) Y_{i(r_i)}^{\alpha_j} (\ln Y_{i(r_i)})^2$$

$$j = 1, \dots, p \quad i = 1, \dots, s. \quad (3.2.3)$$

To find the information matrix note that the density of $Y_{i(\ell)}$ is

$$f(y_{i(\ell)}) = \frac{n_i!}{(\ell-1)!(n_i-\ell)!} \left(\sum_{j=1}^p \lambda_{ij} \alpha_j y_{i(\ell)}^{\alpha_j-1} \right) \left[1 - \exp \left(- \sum_{j=1}^p \lambda_{ij} y_{i(\ell)}^{\alpha_j} \right) \right]^{\ell-1}$$

$$\exp \left[-(n_i - \ell + 1) \left(\sum_{j=1}^p \lambda_{ij} y_{i(k)}^{\alpha_j} \right) \right], 0 \leq y_{i(\ell)} < \infty,$$

$$\ell = 1, \dots, n_i \quad i = 1, \dots, s. \quad (3.2.4)$$

Hence,

$$E(T_i(\alpha_j)) = \sum_{\ell=1}^{r_i} \int_0^{\alpha_j} y_{i(\ell)} f(y_{i(\ell)}) dy_{i(\ell)} + (n_i - r_i) \int_0^{\alpha_j} y_{i(r_i)} f(y_{i(r_i)}) dy_{i(r_i)}, \quad (3.2.5)$$

$$E(T_i^{(1)}(\alpha_j)) = \sum_{\ell=1}^{r_i} \int_0^{\alpha_j} y_{i(\ell)} \ell n y_{i(\ell)} f(y_{i(\ell)}) dy_{i(\ell)} + (n_i - r_i) \int_0^{\alpha_j} y_{i(r_i)} \ell n y_{i(r_i)} f(y_{i(r_i)}) dy_{i(r_i)} \quad (3.2.6)$$

$$E(T_i^{(2)}(\alpha_j)) = \sum_{\ell=1}^{r_i} \int_0^{\alpha_j} y_{i(\ell)} (\ell n y_{i(\ell)})^2 f(y_{i(\ell)}) dy_{i(\ell)} + (n_i - r_i) \int_0^{\alpha_j} y_{i(r_i)} (\ell n y_{i(r_i)})^2 f(y_{i(r_i)}) dy_{i(r_i)},$$

$$i = 1, \dots, s \quad j = 1, \dots, p. \quad (3.2.7)$$

These integrals must be evaluated numerically. Now with r_i fixed, (r_{i1}, \dots, r_{ip}) has a multinomial distribution with parameters π_{ij} given

$$\pi_{ij} = \int_0^{\infty} \lambda_{ij} \alpha_j u^{\alpha_j-1} \exp\left(-\sum_{\ell=1}^p \lambda_{i\ell} u^{\alpha_\ell}\right) du, \quad (3.2.8)$$

so

$$E(r_{ij}) = r_i \pi_{ij}.$$

The information matrix can be obtained by using (3.2.5), (3.2.6), (3.2.7), and (3.2.8) to calculate the expected values of (3.1.7), (3.1.9), and (3.2.10). An estimate of this matrix can be obtained by substituting $\hat{\alpha}_j$ and $\hat{\beta}_j$ in the appropriate expressions.

3.3 Progressive censoring

For this censoring scheme N_i items are put on test at the i^{th} stress level. Let $\tau_{i1}, \dots, \tau_{iM_i}$ be fixed censoring times at which a fixed number of items, c_{i1}, \dots, c_{iM_i} are removed from the test. At time τ_{iM_i} either a fixed number c_{iM_i} items are removed from the test or testing is terminated with a random number, c_{iM_i} , systems still functioning. Assume that N_i is sufficiently large to allow removal of the required number of items.

Let $n_i = N_i - \sum_{k=1}^{M_i} c_{ik}$ be the number of items which are observed to fail and let $Y_{i\ell}$, $\ell = 1, \dots, n_i$ denote their failure times. That is, Y_{i1}, \dots, Y_{in_i} are the n_i system failure times regardless of the mode of failure. Suppose that r_{ij} of the n_i failures were caused by failure of the j^{th} component, and the respective failure times are $X_{ij1}, \dots, X_{ijr_{ij}}$, $i = 1, \dots, s$, $j = 1, \dots, p$.

The likelihood function of interest is given by (3.1.1) which can be factored into component likelihoods as in (3.1.2) with

$$T_i(\alpha_j) = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha_j} + \sum_{\ell=1}^{M_i} \tau_{i\ell}^{\alpha_j} c_{i\ell}, \quad j = 1, \dots, p, \quad i = 1, \dots, s.$$

The likelihood equations are as in (3.1.4), (3.1.5) with

$$T_i^{(1)}(\alpha_j) = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha_j} \ln Y_{i\ell} + \sum_{\ell=1}^{M_i} \tau_{i\ell}^{\alpha_j} c_{i\ell} \ln(\tau_{i\ell}),$$

$$j = 1, \dots, p \quad i = 1, \dots, s. \quad (3.3.2)$$

The matrix of second partial derivatives are as in (3.1.6),

(3.1.7), and (3.1.9) with

$$T_i^{(2)}(\alpha_j) = \sum_{\ell=1}^{n_i} Y_{i\ell}^{\alpha_j} (\ln Y_{i\ell})^2 + \sum_{\ell=1}^{M_i} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell})^2 c_i$$

$$j = 1, \dots, p \quad i = 1, \dots, s. \quad (3.3.3)$$

To calculate $E(T_i(\alpha_j))$, $E(T_i^{(1)}(\alpha_j))$ and $E(T_i^{(2)}(\alpha_j))$ consider any of the s stress levels. The n_i observed failures, $Y_{i\ell}$, have the following survival function

$$F_i(y) = \exp - \left(\sum_{j=1}^p \lambda_{ij} y^{\alpha_j} \right), y \geq 0, i = 1, \dots, s. \quad (3.3.4)$$

Let $f_{i\ell}$ denote the number of failures in the interval $[\tau_{i,\ell-1}, \tau_{i\ell})$, $\ell = 1, \dots, M_i + 1$ where $\tau_{iM_i+1} = \infty$. Define $\bar{F}_{i\ell} = F_i(\tau_{i\ell})$ and $F_{i\ell} = 1 - \bar{F}_{i\ell}$, and let $U_{i\ell k}$, $k = 1, \dots, f_{i\ell}$ denote the failure times of the $f_{i\ell}$ items which fail in this interval.

Cohen (1963) shows that

$$E(f_{i\ell}) = \begin{cases} N\bar{F}_{i\ell} & \text{for } \ell = 1 \\ \left(N - \sum_{k=1}^{\ell-1} \frac{c_{ik}}{F_{ik}} \right) (\bar{F}_{i\ell} - \bar{F}_{i\ell-1}) & \text{for } \ell = 1, \dots, M_i + 1 \end{cases} \quad (3.3.5)$$

if c_{M_i} is fixed and for $\ell = 1, \dots, M_i$ if c_{iM_i} is random

Now

$$E \left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j} \right) = EE \left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j} | f_{i\ell} \right)$$

$$= E(f_{i\ell}) E(U_{i\ell k}^{\alpha_j} | \tau_{i\ell} \leq U_{i\ell k} < \tau_{i\ell+1})$$

$$= E(f_{i\ell}).$$

$$\int_{\tau_{i\ell-1}}^{\tau_{i\ell}} u^{\alpha_j} \left(\sum_{q=1}^p \alpha_q \lambda_{iq} u^{\alpha_q-1} \right) \exp \left(- \sum_{q=1}^p \lambda_{iq} u^{\alpha_q} \right) du / (F_{i\ell} - F_{i\ell-1})$$

for $\ell = 1, \dots, M_i$ (3.3.6)

where $E(f_i)$ is given in (3.3.5) and the integral must be evaluated numerically, with similar expressions for

$E \left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j} \ln U_{i\ell k} \right)$ and $E \left(\sum_{k=1}^{f_{i\ell}} U_{i\ell k}^{\alpha_j} (\ln U_{i\ell k})^2 \right)$. Using these expressions when a fixed number of items are removed at time τ_{iM_i} we have

$$E(T_i(\alpha_j)) = N_i \int_0^\infty u^{\alpha_j} \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du$$

$$- \sum_{\ell=1}^{M_i} c_{i\ell} \int_{\tau_{i\ell}}^\infty u^{\alpha_j} \frac{\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1}}{F_{i\ell}} \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du + \sum_{\ell=1}^{M_i} c_{i\ell} \tau_{i\ell}^{\alpha_j}, \quad (3.3.7)$$

$$E(T_i^{(1)}(\alpha_j)) = N_i \int_0^\infty u^{\alpha_j} \ln u \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du$$

$$+ \sum_{\ell=1}^{M_i} c_{i\ell} \tau_{i\ell}^{\alpha_j} \ln(\tau_{i\ell})$$

$$- \sum_{\ell=1}^{M_i} \frac{c_{i\ell}}{F_{i\ell}} \int_{\tau_{i\ell}}^\infty u^{\alpha_j} \ln u \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du$$

(3.3.8)

and

$$\begin{aligned}
 E(T_i^{(2)}(\alpha_j)) = & N_i \int_0^\infty u^{\alpha_j} (\ln u)^2 \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \\
 & - \sum_{\ell=1}^{M_i} \frac{c_{i\ell}}{F_{i\ell}} \int_{\tau_{i\ell}}^\infty u^{\alpha_j} (\ln u)^2 \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \\
 & + \sum_{\ell=1}^{M_i} c_{i\ell} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell})^2 \\
 & i = 1, \dots, s \quad j = 1, \dots, p. \quad (3.3.9)
 \end{aligned}$$

When all testing stops at time τ_{iM_i} there are

$$c_{M_i} = N_i - \sum_{\ell=1}^{M_i-1} c_{i\ell} - \sum_{\ell=1}^{M_i} f_{i\ell} \text{ items removed from test. Thus}$$

$$E(r_{M_i}) = F_{iM_i} \left(N - \sum_{i=1}^{M_i-1} \frac{c_{i\ell}}{F_{i\ell}} \right).$$

And, here,

$$\begin{aligned}
 E(T_i(\alpha_j)) = & N_i \left\{ \int_0^{\tau_{iM_i}} u^{\alpha_j} \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \right. \\
 & \left. + F_{iM_i} \tau_{iM_i}^{\alpha_j} \right\} \\
 & - \sum_{\ell=1}^{M_i-1} \frac{c_{i\ell}}{F_{i\ell}} \int_{\tau_{i\ell}}^{\tau_{iM_i}} u^{\alpha_j} \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du
 \end{aligned}$$

$$+ \sum_{\ell=1}^{M_i-1} c_{i\ell} \tau_{i\ell}^{\alpha_j} - \sum_{\ell=1}^{M_i-1} c_{i\ell} \frac{F_{iM_i}}{F_{i\ell}} \tau_{iM_i}^{\alpha_j}, \quad (3.3.10)$$

$$E(T_i^{(1)}(\alpha_j)) =$$

$$\begin{aligned} & N_i \left\{ \int_0^{\tau_{iM_i}} u^{\alpha_j} \ln u \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \right. \\ & + F_{iM_i} \tau_{iM_i}^{\alpha_j} \ln \tau_{iM_i} \left. \right\} - \sum_{\ell=1}^{M_i-1} \frac{c_{i\ell}}{F_{i\ell}} \left\{ \int_0^{\tau_{iM_i}} u^{\alpha_j} \ln u \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \right. \\ & \cdot \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du + F_{iM_i} \tau_{iM_i}^{\alpha_j} \ln \tau_{iM_i} \left. \right\} + \sum_{\ell=1}^{M_i-1} c_{i\ell} \tau_{i\ell}^{\alpha_j} \ln \tau_{i\ell}, \end{aligned} \quad (3.3.11)$$

and

$$E(T_i^{(2)}(\alpha_j)) =$$

$$\begin{aligned} & N_i \left\{ \int_0^{\tau_{iM_i}} u^{\alpha_j} (\ln u)^2 \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \right. \\ & + F_{iM_i} \tau_{iM_i}^{\alpha_j} (\ln \tau_{iM_i})^2 \left. \right\} - \sum_{\ell=1}^{M_i-1} \frac{c_{i\ell}}{F_{i\ell}} \left[F_{iM_i} \tau_{iM_i}^{\alpha_j} (\ln \tau_{iM_i})^2 \right. \\ & + \int_{\tau_{i\ell}}^{\tau_{iM_i}} u^{\alpha_j} (\ln u)^2 \left(\sum_{k=1}^p \alpha_k \lambda_{ik} u^{\alpha_k-1} \right) \exp \left(- \sum_{k=1}^p \lambda_{ik} u^{\alpha_k} \right) du \left. \right] \\ & + \sum_{\ell=1}^{M_i-1} c_{i\ell} \tau_{i\ell}^{\alpha_j} (\ln \tau_{i\ell})^2. \end{aligned} \quad (3.3.12)$$

Also when c_{iM_i} is random, n_i is a random variable with mean

$$E(n_i) = NF_{iM_i} - \sum_{l=1}^{M_i-1} c_{il} \left[\frac{F_i - F_{iM_i}}{F_{il}} \right] \quad (3.3.13)$$

Substituting these expectations in the appropriate places in equations (3.1.7) to (3.1.10) yields the asymptotic covariance matrix of $(\hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j}, \hat{\alpha}_j)$ for this censoring scheme.

3.4 Estimation of Parameters at the Usage Stress

Let $\hat{\alpha}_j, \hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j}$ be the maximum likelihood estimators of $\alpha_j, \beta_{j0}, \dots, \beta_{jk_j}$, $j = 1, \dots, p$ obtained from an accelerated life test as described in section 3.1 to 3.3. Let Σ_j denote the covariance matrix of $(\hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j}, \hat{\alpha}_j)$. Recall Σ_j is of the form

$$\Sigma_j = \begin{pmatrix} \Sigma_{(j,j)} & \Sigma_{(j,\alpha)} \\ \Sigma^T_{(j,\alpha)} & \sigma^2_{\alpha,\alpha} \end{pmatrix} \quad (3.4.1)$$

where $\Sigma_{(j,j)}$ is the covariance matrix of $(\hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j})$, $\Sigma_{(j,\alpha)}$ is the vector of covariances between $\hat{\alpha}_j$ and $(\beta_{j0}, \dots, \beta_{jk_j})$ ($\hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j}$) and $\sigma^2_{\alpha,\alpha}$ is the variance of $\hat{\alpha}_j$. Let $\hat{\Sigma}_j$ be an estimator of Σ_j . Let V_u denote the design stress of the system.

For sufficiently large sample sizes the vector $(\hat{\beta}_{j0}, \dots, \hat{\beta}_{jk_j}, \hat{\alpha}_j)$ is approximately normal with mean vector $(\beta_{j0}, \dots, \beta_{jk_j}, \alpha_j)$ and covariance matrix Σ_j . Following Thomas, Bain and Antle (1969) we recommend sample sizes of at least a hundred at each stress level.

At stress V_u the maximum likelihood estimator of the scale parameter of the j th components time to failure distribution, λ_{ju} , is

$$\hat{\lambda}_{ju} = \exp \left(\sum_{\ell=0}^{k_j} \hat{\beta}_{j\ell} \theta_{j\ell}(V_u) \right), \quad j = 1, \dots, p. \quad (3.4.2)$$

For sufficiently large sample sizes $\hat{\lambda}_{ju}$ has a log normal distribution with mean λ_{ju} and variance σ_{ju}^2 given by

$$\sigma_{ju}^2 = (1, \theta_{j1}(V_u), \dots, \theta_{jk_j}(V_u)) \Sigma_{jj} (1, \theta_{j1}(V_u), \dots, \theta_{jk_j}(V_u))^T. \quad (3.4.3)$$

Hence a reduced biased estimator of λ_{ju} is

$$\tilde{\lambda}_{ju} = \hat{\lambda}_{ju} \exp(-\hat{\sigma}_{ju}^2/2), \quad j = 1, \dots, p \quad (3.4.4)$$

where $\hat{\sigma}_{ju}^2$ is obtained by replacing Σ_{jj} by $\hat{\Sigma}_{jj}$ in (3.4.3). If Σ_{jj} were known the mean squared error of $\tilde{\lambda}_{ju}$ is $\lambda_{ju}^2(\exp(\sigma_{ju}^2)-1)$ which is always smaller than $\lambda_{ju}^2(\exp(2\sigma_{ju}^2)-2\exp(\sigma_{ju}^2)+1)$, the mean squared error of $\hat{\lambda}_{ju}$. An asymptotic $(1 - \alpha) \times 100\%$ confidence interval for λ_{ju} is given by

$$(\hat{\lambda}_{ju} \exp(-Z_{1-\alpha/2} \hat{\sigma}_{ju}), \hat{\lambda}_{ju} \exp(Z_{1-\alpha/2} \hat{\sigma}_{ju})), \quad j = 1, \dots, p. \quad (3.4.4)$$

The maximum likelihood estimator of the j^{th} components cumulative hazard rate at stress V_u and time t , $\Lambda_{ju}(t)$ is given by

$$\hat{\Lambda}_{ju}(t) = t^{\hat{\alpha}_j} \hat{\lambda}_{ju}, \quad j = 1, \dots, p, \quad t \geq 0. \quad (3.4.5)$$

This estimator is also biased. An asymptotically unbiased estimator of $\Lambda_{ju}(t)$ is

$$\tilde{\Lambda}_{ju}(t) = \hat{\Lambda}_{ju}(t) \exp(-\hat{\sigma}_j^2(t)/2), \quad j = 1, \dots, p, \quad t > 0 \quad (3.4.6)$$

where

$$\begin{aligned} \hat{\sigma}_j^2(t) &= (1, \theta_{j1}(V_u), \theta_{jk_j}(V_u), \ln t) \\ &\hat{\Sigma}_j(1, \theta_{j1}(V_u), \dots, \theta_{jk_j}(V_u), \ln t)^T. \end{aligned} \quad (3.4.7)$$

This estimator also has smaller mean squared error than $\hat{\Lambda}_{ju}(t)$. Asymptotic $(1 - \alpha) \times 100\%$ confidence intervals for $\Lambda_{ju}^{(t)}$ are given by

$$(\hat{\Lambda}_{ju}(t)\exp(-Z_{1-\alpha/2}\hat{\sigma}_j(t)), \hat{\Lambda}_{ju}(t)\exp(Z_{1-\alpha/2}\hat{\sigma}_j(t))),$$

$$j = 1, \dots, p \quad t > 0. \quad (3.4.8)$$

The maximum likelihood estimator of the j^{th} components survival function at time t and stress V_u is given by

$$\hat{F}_{ju}(t) = \exp(-\hat{\Lambda}_{ju}(t)), \quad j = 1, \dots, p \quad (3.4.9)$$

Approximate $(1 - \alpha) \times 100\%$ confidence intervals for the j^{th} components survival function at time t and stress V_u are given by

$$(\hat{F}_{ju}(t)^{\exp(Z_{1-\alpha/2}\hat{\sigma}_j(t))}, \hat{F}_{ju}(t)^{\exp(-Z_{1-\alpha/2}\hat{\sigma}_j(t))})$$

$$j = 1, \dots, p \quad t \geq 0 \quad (3.4.10)$$

Let κ be a subset of $1, \dots, p$ of cardinality k . We are interested in obtaining estimators of

$$F_u^{(\kappa)}(t) = \prod_{j \in \kappa} F_{ju}(t), \quad (3.4.11)$$

the survival function of an item which can fail only from the failure of components indexed by elements of κ . When $\kappa = \{1, \dots, p\}$ then (3.4.1) is the overall system survival function. When κ is a proper subset of $\{1, \dots, p\}$ then (3.4.1) represents the survival function of a system which has been redesigned so that those components indexed by κ^c are extremely reliable. Clearly, the maximum likelihood estimator of $F_u^{(\kappa)}(t)$ is

$$\hat{F}^{(\kappa)}(t) = \prod_{j \in \kappa} \hat{F}_{ju}(t). \quad (3.4.12)$$

An approximate $(1 - \alpha) \times 100\%$ confidence interval for $F_u^{(\kappa)}(t)$ is

$$\left(\prod_{j \in K} \hat{F}_{ju}^{\exp(\sigma_{ju} Z_\gamma)}, \prod_{j \in K} \hat{F}_{ju}^{\exp(-\sigma_{ju} Z_\gamma)} \right) \quad (3.4.13)$$

where $\gamma = \frac{1+(1-\alpha)^{1/k}}{2}$. This is a conservative interval in the following sense. From (3.4.10)

$$(1-\alpha)^{1/k} = P(\hat{F}_{ju}(t)^{\exp(Z_r \hat{\sigma}_j(t))} \leq F_{ju}(t) \leq \hat{F}_{ju}(t)^{\exp(-Z_r \hat{\sigma}_j(t))}) \text{ for } j \in K.$$

Since $(\hat{F}_{ju}(t), \hat{F}_{j'u}(t))$ are asymptotically independent for $j \neq j'$,

$$(1-\alpha) = P(\hat{F}_{ju}(t)^{\exp(Z_r \hat{\sigma}_j(t))} \leq F_{ju}(t) \leq \hat{F}_{ju}(t)^{\exp(-Z_r \hat{\sigma}_j(t))}) \text{, for all } j \in K)$$

$$\leq P\left(\prod_{j \in K} \hat{F}_{ju}(t)^{\exp(Z_r \hat{\sigma}_j(t))} \leq F_{ju}(t) \leq \prod_{j \in K} \hat{F}_{ju}(t)^{\exp(-Z_r \hat{\sigma}_j(t))}\right).$$

3.5 Dependent Risks

In sections 3.1 - 3.4 it was assumed that the component lifetimes were independent. This assumption may be relaxed by considering a fatal shock model. For simplicity we shall illustrate this model for the bivariate case.

Let U_1, U_2, U_{12} be independent Weibull random variables with shape parameters $\alpha_1, \alpha_2, \alpha_{12}$ and scale parameters $\lambda_1(V, \underline{\beta}_1), \lambda_2(V, \underline{\beta}_2),$ and $\lambda_{12}(V, \underline{\beta}_{12})$ in an environment characterized by a constant application of a stress V . Here U_1 represents the time until a shock destroys the first component only, U_2 the time until a shock destroys the second component, and U_{12} the time until a shock destroys both components. If (X_1, X_2) represent the component lifetimes then, clearly, $X_1 = \min(U_1, U_{12})$, and $X_2 = \min(U_2, U_{12})$. The component survival functions are not Weibull, but are given by

$$F_j(t; V) = \exp(-\lambda_j(V, \underline{\beta}_j) t^{\alpha_j} - \lambda_{12}(V, \underline{\beta}_{12}) t^{\alpha_{12}}),$$

$$j = 1, 2 \quad t \geq 0. \quad (3.5.1)$$

An accelerated life test can be conducted as before where now the parameters of interest are α_1 , α_2 , α_{12} , β_1 , β_2 , β_{12} . The results of section 3.4 can be used to obtain estimators of the component survival function under normal conditions.

4. EXAMPLE

As an example of these procedures we shall consider an example given in Nelson (1974a). The problem is to analyze an accelerated life test conducted on Class-H insulation systems for electric motors. There are three possible types of insulation failures corresponding to distinct parts of the insulation system, namely Turn, Phase, and Ground. The failure cause is determined by an engineering examination of the failed motor.

The purpose of the experiment is to estimate the average life of such insulation systems at a design temperature of 180° C. A median life of 20,000 hours is necessary for the satisfactory performance of these insulation systems. To reduce test time and cost an accelerated life test was conducted at 4 accelerated temperatures, namely, 190° C, 220° C, 240° C, and 260° C.

The accelerated life test was conducted by putting 10 motors on test at each of the 4 stress levels. Motors were run until they failed, then the cause of failure was found and isolated and motors were run until a second failure occurred. The results of this study are reported in Nelson (1974a). The data followed a \log_{10} normal distribution so the Weibull theory results do not apply.

To illustrate the results of the previous section Nelson's example is reproduced by simulating the life test using a Weibull model with shape parameter 1 for each failure cause. The shift parameters are chosen by fitting an Arrhenius Reaction Rate model to the estimated component medians obtained by Nelson. The model is

Table 4.1

Life Test Data Simulated From Nelson's Example

Motor	180 Degrees		190 Degrees		220 Degrees		240 Degrees	
	Failure Time	Cause	Failure Time	Cause	Failure Time	Cause	Failure Time	Cause
1	5606.0781	Turn	1628.8145	Ground	344.1240	Phase	557.4395	Ground
2	4905.0859	Turn	1097.6609	Turn	761.8518	Phase	156.9030	Phase
3	2871.9370	Phase	630.0374	Phase	1562.7520	Turn	906.5212	Phase
4	2762.9712	Ground	1520.8772	Ground	276.9924	Ground	61.1210	Turn
5	3413.8027	Turn	708.5212	Phase	482.2432	Turn	773.3906	Turn
6	6321.7617	Turn	205.9655	Phase	213.3295	Turn	148.7977	Ground
7	4847.3906	Turn	185.6579	Turn	1434.3723	Turn	41.1974	Turn
8	2690.2847	Turn	434.2930	Turn	1486.6152	Turn	787.6323	Turn
9	38.9871	Phase	1938.7297	Phase	1355.4917	Turn	224.2534	Ground
10	2358.2275	Phase	3093.8237	Turn	1374.0374	Phase	405.3303	Ground
11	3755.4043	Ground	1171.8782	Ground	725.5413	Turn	1071.6702	Ground
12	4898.8477	Phase	1108.7510	Ground	917.9756	Ground	407.1978	Turn
13	3900.2310	Turn	27.5321	Turn	2970.2925	Ground	306.0037	Ground
14	1196.4922	Turn	1428.3220	Turn	609.9128	Turn	422.7825	Turn
15	6000.6875	Ground	263.7917	Phase	89.8835	Turn	178.5943	Turn
16	1645.1550	Ground	1113.6123	Turn	741.6179	Turn	588.4976	Turn
17	4021.5698	Ground	965.0088	Ground	706.0217	Ground	301.6204	Phase
18	2643.6931	Turn	49.1324	Phase	347.2078	Turn	14.8288	Turn
19	4760.6289	Turn	350.6594	Turn	238.5782	Phase	1315.0032	Ground
20	1621.5481	Turn	2026.7441	Phase	1001.3643	Ground	90.0674	Turn

$$\lambda_i(V; \underline{\beta}_j) = \exp(\beta_{j0} + \beta_{j1}\theta_{j1}(V)), \quad j = 1, 2, 3 \quad (4.1)$$

where $\theta_{j1}(V) = -1000/V$ for $j = 1, 2, 3$ and V is the temperature in degrees absolute. The absolute temperature is 273.16 plus the centigrade temperature. The values of (β_{j0}, β_{j1}) , $j = 1, 2, 3$ are as follows:

Table 4.2 True Values of β_0, β_1

	β_0	β_1
Turn	8.2607	8.0106
Phase	3.7748	6.1253
Ground	13.0340	10.6487

Twenty Weibull observations were generated at each of the four stress levels. The data are in Table 4.1.

A Newton-Raphson procedure was used to solve the likelihood equations. The integrals in (3.1.14), (3.1.15), (3.1.16), and (3.1.17) were evaluated using a repeated seven point Gauss-Laguerre formula. The maximum likelihood estimates are as follows:

$$\begin{aligned} \text{TURN: } \hat{\alpha} &= 1.0099, \hat{\beta}_0 = 6.1363, \hat{\beta}_1 = 6.9390 \\ \text{PHASE: } \hat{\alpha} &= .9993, \hat{\beta}_0 = 3.5831, \hat{\beta}_1 = 6.0272 \\ \text{GROUND: } \hat{\alpha} &= 1.0395, \hat{\beta}_0 = 7.9788, \hat{\beta}_1 = 8.2679. \end{aligned}$$

The estimated covariance matrices are:

$$\begin{aligned} \hat{\Sigma}_{\text{TURN}} &= \begin{pmatrix} 9.8252 & 5.3254 & .1178 \\ 5.3254 & 3.1306 & .1283 \\ .1178 & .1283 & .0188 \end{pmatrix} \\ \hat{\Sigma}_{\text{PHASE}} &= \begin{pmatrix} 21.7658 & 11.6281 & .2217 \\ 11.6281 & 6.6803 & .2442 \\ .2282 & .2442 & .0353 \end{pmatrix} \\ \hat{\Sigma}_{\text{GROUND}} &= \begin{pmatrix} 18.2378 & 9.8173 & .1856 \\ 9.8173 & 5.6782 & .2033 \\ .1856 & .2033 & .0299 \end{pmatrix} \end{aligned}$$

Using these estimates, at a design stress of 180% the estimates of the probability of component survival at a mission time of 20,000 hours is .1020 for turn failures, .3023 for phase failures, and .3574 for ground failures. 90% confidence intervals for the probability of component survival at 20,000 hours and a temperature of 180° C are:

TURN - (.0187, .2701),
PHASE - (.0749, .5756),
GROUND - (.1191, .6080).

Using equations (3.4.12) and (3.4.13) the maximum likelihood estimate and 90% confidence interval for system reliability at 20,000 hours and a temperature of 180° C are .0110 and (.000027, .1402). Similarly, a 90% confidence interval for a redesigned system in which turn failures cannot occur is (.0043, .4024).

We note that the above confidence intervals are suspect due to the relatively small sample sizes and are provided here to only to illustrate this procedure.

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BIBLIOGRAPHY

- Armitage, P. and Doll, R. (1961). Stochastic models for carcinogens. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. 19-38.
- Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, 5, 327-339.
- David, H. A. and Moeschberger, M. L. (1978). *The Theory of Competing Risks*. New York: MacMillan Publishing Co., Inc.
- Hartley, H. O. and Sielken, R. L. (1977). Estimation of 'safe dose' in carcinogenics. *Biometrics*. 33, 1-30.

- Kielpinski, T. J. and Nelson, W. B. (1975). Optimal censored accelerated life tests for normal and log-normal life distributions. *IEEE Transactions on Reliability*, V-R-24, 310-320.
- Klein, J. P. and Basu, A. P. (1980a). Accelerated life testing under competing exponential failure distributions. (Submitted for publication).
- Klein, J. P. and Basu, A. P. (1980b). Accelerated life tests under competing Weibull causes of failure. (Submitted for publication).
- Mann, N. R. (1978). Calculation of small-sample Weibull tolerance bounds for accelerated testing. *Comm. Statist.-Theor. Meth.* A7, 97-112.
- Mann, N. R. (1972). Design of over-stress life-test experiments when failure times have a two-parameter Weibull distribution. *Technometrics*. 14, 437-451.
- Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Test Data*. New York: John Wiley & Sons, Inc.
- Meeker, W. Q. and Nelson, W. B. (1974a). Charts for optimal tests for the Weibull and extreme value distributions. *General Electric Company, Corporation Research and Development Technical Information Series Report*. 74-CRD-274.
- Meeker, W. Q. and Nelson, W. B. (1974b). Theory for optimal accelerated tests for Weibull and extreme value distributions. *General Electric Company, Corporation Research and Development Technical Information Series Report*. 74-CRD-248.
- Nelson, W. B. (1970). Statistical methods for accelerated life test data - the inverse power law model. *General Electric Company, Corporation Research and Development Technical Information Service Report*. 71-C-011.
- Nelson, W. B. (1971). Analysis of accelerated life test data - part I: the Arrhenius model and graphical methods. *IEEE Transactions on Electrical Insulation*. VEI-6, 165-181.
- Nelson, W. B. (1972a). Analysis of accelerated life test data - part II: numerical methods and test planning. *IEEE Transactions on Electrical Insulation*. VIE-7, 36-55.

- Nelson, W. B. (1972b). Analysis of accelerated life test data - part III: product comparison and checks on the validity of the model and data. *IEEE Transactions on Electrical Insulation*. VEI-7, 99-119.
- Nelson, W. B. (1972c). Graphical analysis of accelerated life test data with the inverse power law model. *IEEE Transactions on Electrical Insulation*. VEI-21, 2-11; Correction VR-21, 295.
- Nelson, W. B. (1973). Graphical analysis of accelerated life test data with different failure modes. *General Electric Company, Corporation Research and Development Technical Information Series Report*. 73-CRD-001.
- Nelson, W. B. (1974b). Analysis of accelerated life test data with a mix of failure modes by maximum likelihood. *General Electric Company, Corporation Research and Development Technical Information Series Report*. 74-CRD-160.
- Nelson, W. B. (1974a). Methods for planning and analyzing accelerated tests. *General Electric Company, Corporation Research and Development Technical Information Series Report*. 73-CRD-034.
- Nelson, W. B. and Hahn, G. J. (1972). Regression analysis for censored data - part I: simple methods and their applications. *Technometrics*. 14, 247-269.
- Nelson, W. B. and Hahn, G. J. (1973). Regression analysis for censored data - part II: best linear unbiased estimation and theory. *Technometrics*. 15, 133-150.
- Thomas, D. R., Bain, L. J. and Antle, C. E. (1969). Inferences on the parameters of the Weibull distribution. *Technometrics*. 11, 445-460.